Quqn 203 - 2021

Qn ~~one~~ I.

The value of $\theta$ that maximizes $L(\theta)$ if it exists is some $\theta$ such that $L(\theta^*) \geq L(\theta)$.

Since log is increasing.

$L(\theta^*) \geq L(\theta)$ this implies that $\log(\theta^*) = \log(\theta)$

i.e. $\ln(\theta)$.

Maximising $\ln(\theta)$ involves $\frac{d}{\partial\theta}(\ln(\theta)$ and setting it equal to zero and

maximising $L(\theta)$ involve taking the derivative of $L(\theta)$ and setting it equal to zero

$$\frac{\partial}{\partial\theta}\ln(\theta) = \frac{L'(\theta)}{L(\theta)}.$$

Thus $\frac{\partial}{\partial x}\ln(\theta) = 0 = \frac{L'(\theta)}{L(\theta)} = 0$

Thus $L'(\theta) = 0$ in both cases showing that their maximizer is the same.

Qn 2

(a) $\int x^p(1-x)\,dx$.

$\int x^p - x^{p+1}\,dx$.

$= \dfrac{x^{p+1}}{p+1} - \dfrac{x^{p+2}}{p+2} + C$

L.C.M $= (p+1)(p+2)$

~~constant of proportionality is $(p+1)(p+2)$.~~

$$\frac{(p+2)x^{p+1} - (p+1)x^{(p+2)}}{\cancel{(p+1)(p+2)}}$$

Constant of Proportionality is $(p+1)(p+2)$

(b) $\int x(x^p(1-x)\,dx$.

$\int x^{p+1}(1-x)\,dx$.

$\int x^{p+1} - \int x^{p+2}\,dx$

$$\frac{x^{p+2}}{p+2} - \frac{x^{p+3}}{p+3} = \frac{p+1}{p+3}$$

(c) $$\ln = \prod_{i=1}^{n} x^p(1-x).$$

$$\ln = \frac{\prod^{n}\left(x^{p+1} - x^{p+2}\right)}{\prod(p+1)\,x_p^n}$$

introducing logarithms.

$\ln(Lp) = n\ln(p+1) + n\ln(p+2) + p\sum \ln x_i + \sum \ln(1-x_i)$

Qn 3

(a) $f(x) = \theta x^{\theta-1}, \quad 0<x<1 \quad \theta>0$

$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

factor $\theta$ we have

$L(\theta) = \hat{\theta}\left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

Introducing natural log

$\ln(L(\theta)) = \ln \theta^n \cdot \left(\prod x_i\right)^{\theta-1}$

$= n\ln\theta + (\theta-1)\ln\left(\prod_{j=1}^{n} x_i\right)$

$= n\ln\theta + (\theta-1)\sum \ln x_i$

MLE is calculated by taking derivative of $\theta \ln(L)(\theta)$.

$\left[n\ln(\theta) + (\theta-1)\ln\left(\prod_{j=1}^{n} x_i\right)\right]' = \frac{n}{\theta} + \ln\left(\prod_{j=1}^{n} x_i\right)$

$\Rightarrow \theta_0 = \frac{-n}{\ln\left(\prod_{i=1}^{n} x_i\right)} = \frac{-1}{\ln(x)}$ is the MLE of $\theta$

(b) $f(x) = \theta x^{\theta-1}; \quad 0<x<1$

$Ex = \int_0^1 x \cdot f_x \, dx$

$= \int_0^1 x \cdot \theta x^{\theta-1} dx$

$\theta \int_0^1 x \cdot x^{\theta-1} dx$

$= \theta \int_0^1 x \, dx$

$\theta \int_0^1 x^{\theta} dx$

$\theta\left[\frac{x^{\theta+1}}{\theta+1}\right] = \theta\left[\frac{1}{\theta+1}\right] =$

$\frac{\theta}{\theta+1}$

(c) $ML \neq MM$ i.e Maximum likelihood estimator is not equal to Methods of moment estimator'

(4)

(a) $fx = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \exp(-\lambda x)$

$$\ln(v) = \prod_{i=1} \frac{\lambda^r}{\Gamma(r)} x_i^{r-1} \exp(-\lambda x_i).$$

$$= \left(\frac{\lambda^r}{\Gamma(r)}\right)^n \prod_{i=1}^{n} x_i^{r-1} \exp(-\lambda x_i)$$

$$= \left(\frac{\lambda^r}{\Gamma(r)}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{r-1} \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

droping the term that does not contain r

The log likelihood is ~~In~~

$$\ln(r) = nr \log\lambda - n \log\Gamma(r) + (r-1) \log\left(\prod_{j=1}^{n} x_i\right)$$

(b) $M(t) = \left(\frac{\lambda}{\lambda+t}\right)^r$

$$\text{Mean} = \frac{d}{d\lambda}(mt) = \frac{d}{d\lambda}\left(\frac{\lambda}{\lambda-t}\right)^r = \cancel{x}\left(\lambda(\lambda-t)^{-1}\right)^{r-1}\left(\cancel{r}+(\lambda-t)^{-1}\right)\Big|_{t_0}$$

$$r\left(\lambda(\lambda-t)^{-1}\right)^{r-1}\left(\lambda+r\lambda-t)^{-1}\right)\Big|_{t=0} = r$$

$$\frac{dt^2}{d\lambda^2} M(t) = \frac{d}{dx}\left(r(\lambda(\lambda-t)^{-1})^{r-1}(\lambda+(\lambda-t)^{-1}\right)$$

$=$

$$r(r-1)\left(\lambda(\lambda-t)^{-1}\right)^{r-2}\left(\lambda+(\lambda-t)^{-1}\left(\lambda+(\lambda-t)^{-1}\right)\right)+$$

$$\left(1-1(\lambda-t)^{-2}\right)\left(r\left(\lambda(\lambda-t)^{-1}\right)^{r-1}\right|_{t=0}$$

$$=\left(r-\frac{r}{\lambda^2}\right)\left(r^2-r\right) = r^3-r^2-\frac{r^3}{\lambda^2}+\frac{r^2}{\lambda^2}$$

$$\text{Variance} = r^3-r^2-\frac{r^3}{\lambda^2}+\frac{r^2}{\lambda^2}-(r)^2$$

$$= r^3-2r^2-r^3-\left(\frac{r^3-r^2}{\lambda^2}\right)$$

The maximum likehood estimators are not equal ($\neq$).